IDENTIFICATION OF LINEAR ERROR-MODELS WITH PROJECTED DYNAMICAL SYSTEMS

P. Krejci, Mathematical Institute of the Academy of Sciences, Prague, Czech Republic
K. Kuhnen, Laboratory for Process Automation, Saarland University, Germany
Corresponding Author: K. Kuhnen
Laboratory for Process Automation, Saarland University,
(Dir. Prof. Dr.-Ing. habil. H. Janocha)
Building 13, 66123 Saarbrücken, Germany
Phone: 0049-681/302-4715, Fax: 0049-681/3022678
email: k.kuhnen@lpa.uni-saarland.de

Abstract. Linear error models are an integral part of several parameter identification methods for feedforward and feedback control systems and lead in connection with the $L_2$-norm to a convex distance measure which has to be minimised for identification purposes. The parameters are hereby often subject to specific restrictions whose intersections span a convex solution set with non-differentiability points on its boundary. For solving these well conditioned problems on-line the paper formulates the solution of the bounded convex minimisation problem as a stable equilibrium set of a proper system of differential equations. The vector field of the corresponding system of differential equations is based on a projection of the negative gradient of the distance measure. A general drawback of this approach is the discontinuous right-hand side of the differential equation caused by the projection transformation. The consequence are difficulties for the verification of the existence, uniqueness and stability of a solution trajectory. Therefore the main subject of this paper is the derivation of an alternative formulation of the projected dynamical system, which exhibits, in contrast to the original formulation, a continuous right-hand side and thus is accessible to conventional analysis methods. For this purpose the multi-dimensional stop operator is used. Finally the existence, uniqueness and stability properties of the solution trajectories are established.

1. Introduction

In general the design of control systems bases on a more or less precise model of the control plant. Frequently a model structure for the characteristic of the plant is given with fixed but unknown model parameters. A central step for the design of feedforward or feedback controllers consists in the identification of the unknown parameters of the plant based on measurements of the characteristic [3,6]. The starting point is the definition of an error signal $e$ which describes the deviation of the model characteristic from the measured characteristic in dependence on the unknown model parameters $w$ for every time $t$. The next step consists in the derivation of a measure $V$ for the distance between the modeled and the measured data from the error signal $e$ by using a well suited cost functional. Proper error model parameters are then determined by the minimisation of this distance measure $V$.

An important task which consists in obtaining a practically well-posed identification problem is to find an error model of the form

$$e(t) = \zeta(t) + \Psi^T(t) \cdot w$$

with linear dependence on the parameters. Here, $\zeta$ and $\Psi$ are a scalar and a vector time function which contain the unparametrised characteristic of the plant. Starting from this error model the square of the $L_2$-norm leads to

$$V(w) = \frac{1}{2} w^T \cdot \int_{t_0}^{t_2} \Psi(t)^T \Psi(t)^T dt \cdot w + \int_{t_0}^{t_2} \zeta(t)^T \Psi(t)^T dt \cdot w + \frac{1}{2} \int_{t_0}^{t_2} \zeta(t)^T dt = \frac{1}{2} w^T \cdot H \cdot w + g^T \cdot w + f$$

which is a quadratic distance measure for the identification of the error model parameters $w$ on the measurement time interval $[t_0,t_2]$. By construction, the matrix $H$ given by

$$w^T \cdot H \cdot w = \int_{t_0}^{t_2} (\Psi^T(t) \cdot w)^2 dt \succeq 0 \quad \forall w \in \mathbb{R}^r,$$
is always symmetric and positive-semidefinite. From this follows the convexity of the distance measure $V(w)$. In many cases the parameter set consists only of a convex subset $Z$ of $\mathbb{R}^n$ which can for instance be given in terms of a concave function $u$ as

$$Z = \{ w \in \mathbb{R}^n \mid u(w) \geq 0 \}. \quad (4)$$

A typical example for such an identification problem is the compensator design for memoryless and hysteretic nonlinearities with the so-called Prandtl-Ishlinskii approach. In this approach the conditions for the invertibility of the corresponding nonlinearities are then formulated as linear inequality constraints for the error model parameters [6], and the convex subset $Z$ is therefore a convex polyhedron which results from the intersection of convex halfspaces defined by linear inequality constraints. In this case the optimisation problem to solve is given by

$$\min_{w \in Z \cap \Omega} \{ V(w) \}. \quad (5)$$

Because of the convexity of the distance measure $V$ and the parameter set $Z$ the bounded optimisation problem (5) is convex itself and thus the set of global minima is convex, too [8]. For solving these well conditioned problems before putting the control system into operation the optimisation theory provides multiple powerful algorithms. But these algorithms are not suitable for an optimisation during the operation of the control system, because they require too much computing power. A possibility to avoid these difficulties is to formulate the solution of the bounded quadratic optimisation problem as a stable equilibrium point of a proper system of differential equations. This dynamical system can then be solved through numerical integration very efficiently from time step to time step during the operation of the control system. In the unconstrained case, for instance, the right-hand side of the differential equation is given by the negative gradient of the quadratic target function. The theory of the projected dynamical systems [7] offers a starting point for the formulation of a respective differential equation if the convex constrained $Z$ is present. Then the vector field of the differential equation is also based on the negative gradient of the quadratic target function. However, the right-hand side of the differential equation is obtained in this case from a projection of the negative gradient which ensures that the trajectory of the system under no circumstances leaves the admissible solution set. In contrast to parameter projection methods normally used in the field of adaptive systems, the applied projection transformation considers also the non-differentiability points of the boundaries of the convex solution set. As in the case of a convex polyhedron these non-differentiability points often result from the intersection of smooth convex sets. The projection transformation leads to a discontinuous right-hand side of the differential equation. The consequence are difficulties for the verification of the existence, uniqueness and stability of a solution trajectory. The main subject of this paper is the derivation of an alternative formulation of the projected dynamical system, which exhibits, in contrast to the original formulation, a continuous right-hand side and thus is accessible to conventional analysis methods. For this purpose the multi-dimensional stop operator, well-known from the hysteretic systems theory, is used [4,5]. Finally the existence and uniqueness of the solution trajectories are verified and the stability properties of the projected dynamical system are investigated.

2. Geometry of convex sets

We consider the vector space $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$, endowed with a scalar product $\langle \cdot , \cdot \rangle$ and with norm

$$\| x \| = \langle x , x \rangle^{1/2} \quad (6)$$

for $x \in X$, and a fixed closed convex set $Z \subset X$. For $x \in Z$ we define the outward normal cone $N(x)$ and the tangential cone $T(x)$ by the formula

$$N(x) = \{ y \in X \mid \langle y , x - z \rangle \geq 0 \ \forall z \in Z \} \quad (7)$$

and

$$T(x) = \{ v \in X \mid \langle y , v \rangle \leq 0 \ \forall y \in N(x) \}. \quad (8)$$

Note that we have
\[ z - x \in T(x) \quad \forall x, z \in Z. \quad (9) \]

We denote by \( Q \) the orthogonal projection of \( X \) onto \( Z \), that is,
\[
Q(u) \in Z, \quad \|u - Q(u)\| = \text{dist}(u, Z) \quad (10)
\]
for \( u \in X \). We also make systematic use of the dual mapping \( P(u) = u - Q(u) \). In particular (10) can be equivalently written as a variational inequality
\[
Q(u) \in Z, \quad \langle P(u), Q(u) - z \rangle \geq 0 \quad \forall u \in X \quad \forall z \in Z. \quad (11)
\]

Indeed, this follows from the elementary identity
\[
\|u - ((1 - \alpha)Q(u) + \alpha z)\|^2 = \|u - Q(u)\|^2 + \alpha^2 \|Q(u) - z\|^2 + 2\alpha \langle P(u), Q(u) - z \rangle \quad (12)
\]
for all \( \alpha \in [0,1] \), \( u \in X \), and \( z \in Z \). Assuming that (11) holds, we obtain from (12) for \( \alpha = 1 \) that \( \|u - z\|^2 \geq \|u - Q(u)\|^2 \) for all \( z \in Z \), hence (10) is verified. Conversely, if (10) holds, then \( \|u - ((1 - \alpha)Q(u) + \alpha z)\| \geq \|u - Q(u)\| \)
and from (12) it follows that \( \alpha \langle P(u), Q(u) - z \rangle + 2\langle P(u), Q(u) - z \rangle \geq 0 \) for all \( \alpha \in [0,1] \). Letting \( \alpha \) tend to 0+ we obtain (11). The tangential cone \( T(x) \) is obviously a convex closed set, we thus can define in the same way the projection \( Q_{T(x)} \) of \( X \) onto \( T(x) \) for \( x \in Z \) analogously to (10) as
\[
Q_{T(x)}(u) \in T(x), \quad \|u - Q_{T(x)}(u)\| = \text{dist}(u, T(x)) \quad (13)
\]
for \( u \in X \). The relationship between \( Q \) and \( Q_{T(x)} \) will be characterized in the following two Lemmas.

**Lemma 2.1** Let \( x \in Z \) and \( u, v \in X \) be given. Then
\[
v = Q_{T(x)}(u) \iff v \in T(x), \quad \langle u - v, v \rangle = 0, \quad x = Q(x + u - v). \quad (14)
\]

**Proof.** Assume first that \( v = Q_{T(x)}(u) \). The variational formulation analogous to (11) reads
\[
\langle u - v, v - s \rangle \geq 0 \quad \forall s \in T(x). \quad (15)
\]
As \( T(x) \) is a cone, we can choose consecutively \( s = o \) and \( s = 2v \in T(x) \) and obtain \( \langle u - v, v \rangle = 0 \) and
\[
\langle u - v, -s \rangle \geq 0 \quad \forall s \in T(x). \quad (16)
\]
By (9) we can put \( s = z - x \) for an arbitrary \( z \in Z \), hence
\[
\langle u - v, x - z \rangle \geq 0 \quad \forall z \in Z \quad (17)
\]
which is equivalent to \( x = Q(x + u - v) \) according to (11). Conversely, if the right-hand side of (14) is satisfied and if \( s \in T(x) \) is arbitrary, then
\[
\langle u - v, x - Q(x + s) \rangle \geq 0 \quad \forall \delta > 0, \quad (18)
\]
or equivalently
\[
\langle u - v, \frac{1}{\delta} P(x + \delta s) - s \rangle \geq 0 \quad \forall \delta > 0. \quad (19)
\]
The assertion will immediately follow provided we check that
\[
\lim_{\delta \to 0^+} \frac{1}{\delta} P(x + \delta s) = 0 \quad \forall x \in Z \quad \forall s \in T(x). \quad (20)
\]
To prove this conjecture, we use (11) for \( u = x + \delta s \) and obtain
\[
\langle P(x + \delta s), Q(x + \delta s) - z \rangle \geq 0 \quad \forall z \in Z \quad \forall \delta > 0. \quad (21)
\]
For \( z = x \) this yields in particular
\[
\langle P(x + \delta s), \delta s - P(x + \delta s) \rangle \geq 0 \quad \forall \delta > 0, \quad (22)
\]
hence \(|P(x + \delta s)| \leq \delta |s|\). The system \( \{P(x + \delta s)/\delta; \ \delta > 0\} \) is bounded, we can therefore extract a sequence \( \delta_n \to 0^+ \) and an element \( y \in X \) such that
\[
\lim_{n \to \infty} \frac{1}{\delta_n} P(x + \delta_n s) = y. \quad (23)
\]
Dividing (21) for \( \delta = \delta_n \) by \( \delta_n \) and letting \( n \) tend to \( \infty \) we obtain
\[
\langle y, x - z \rangle \geq 0 \quad \forall z \in Z, \quad (24)
\]
hence \( y \) belongs to the normal cone \( N(x) \). We further divide (22) for \( \delta = \delta_n \) by \( \delta_n^2 \) and pass to the limit. This yields
\[
\langle y, s - y \rangle \geq 0. \quad (25)
\]
By definition of \( N(x) \) and \( T(x) \) we have \( \langle y, s \rangle \leq 0 \), hence \( y = o \) independently of the sequence \( \{\delta_n\} \), and the proof is complete.

**Lemma 2.2**  *For every \( u \in X \) and \( x \in Z \) we have*
\[
Q_{T(x)}(u) = \lim_{\delta \to 0^+} \frac{1}{\delta} (Q(x + \delta u) - x).
\]

*Proof.* Let \( u \in X \) and \( x \in Z \) be given. Similarly as in (21) we have
\[
\langle P(x + \delta u), Q(x + \delta u) - x \rangle = \langle x + \delta u - Q(x + \delta u), Q(x + \delta u) - x \rangle \geq 0 \quad \forall \delta > 0, \quad (26)
\]
hence \(|Q(x + \delta u) - x| \leq \delta |u| \) for every \( \delta > 0 \). There exists again a sequence \( \delta_n \to 0 \) and an element \( v \in X \) such that
\[
\lim_{n \to \infty} \frac{1}{\delta_n} (Q(x + \delta_n u) - x) = v. \quad (27)
\]
For every \( y \in N(x) \) and \( \delta > 0 \) we have by definition of \( N(x) \) that \( \langle y, x - Q(x + \delta u) \rangle \geq 0 \). Dividing this inequality by \( \delta = \delta_n \) by \( \delta_n \) and passing to the limit we obtain that \( \langle y, v \rangle \leq 0 \) for every \( y \in N(x) \), hence \( v \in T(x) \). To check that \( v = Q_{T(x)}(u) \), we choose an arbitrary \( s \in T(x) \) and compute
where
\[ A_s = \frac{1}{\delta_n^2} \langle P(x + \delta_n u), Q(x + \delta_n u) - Q(x) \rangle = A_s - B_s, \]
and
\[ B_s = \frac{1}{\delta_n^2} \langle P(x + \delta_n u), P(x + \delta_n s) \rangle \]
with \( \lim_{n \to \infty} B_n = 0 \) by virtue of (20) and (27). Passing to the limit in (28) we obtain (15), hence \( v = Q_T(u) \). The independence of the limit in (27) from the choice of \( \delta_n \) completes the proof.

Let us consider now a convex continuously differentiable function \( F: X \to \mathbb{R} \), and let \( F': X \to X \) denote its gradient, that is,
\[
\langle F'(x), u \rangle = \lim_{\delta \to 0^+} \frac{1}{\delta} (F(x + \delta u) - F(x)) \text{ for } x, u \in X.
\]

Let
\[
K = \{v \in Z | F(v) \leq F(z) \ \forall z \in Z\}
\]
denote the set (possibly empty if \( Z \) is unbounded and \( F \) is unbounded from below) where \( F \) attains its minimum on the set \( Z \). For \( v_0, v_1 \in K \) we have indeed \( F(v_0) = F(v_1) : F_{\min} \) and for \( \alpha \in [0,1] \) it follows that
\[
F_{\min} \leq F(\alpha v_1 + (1-\alpha) v_0) \leq \alpha F(v_1) + (1-\alpha) F(v_0) = F_{\min},
\]
hence \( K \) is convex. We conclude this section with the following easy and classical result.

**Lemma 2.3** Let \( F, K \) be as above, and set
\[
\hat{K} = \{v \in Z | \langle F'(v), z - v \rangle \geq 0 \ \forall z \in Z\},
\]
\[
K_\alpha = \{v \in Z | a = Q_{T(v)}(-F'(v))\}.
\]

Then \( K_\alpha = \hat{K} = K \).

**Proof.** For \( v \in K \) and \( z \in Z \) we have \( (F(v + \delta(z - v)) - F(v))/\delta \geq 0 \), and letting \( \delta \) tend to 0+ we obtain \( (F'(v), z - v) \geq 0 \), hence \( K \subset \hat{K} \). To prove the inclusion \( \hat{K} \subset K \), we first notice that the convexity of \( F \) yields
\[
F(v + \delta(z - v)) - F(v) \leq \delta(F(z) - F(v)) \ \forall z, v \in Z \ \forall \delta \in [0,1].
\]
For \( \delta \to 0^+ \) this implies that
\[
\langle F'(v), z - v \rangle \leq F(z) - F(v) \ \forall z, v \in Z.
\]
For all \( v \in \hat{K} \) we thus have \( F(z) - F(v) \geq 0 \) for all \( z \in Z \), hence \( v \in K \). The identity \( K_\alpha = \hat{K} \) is straightforward. By Lemma 2.1 we have
\[
a = Q_{T(v)}(-F'(v)) \iff v = Q(v - F'(v))
\]
which is by virtue of (11) in turn equivalent to the variational inequality.
and Lemma 2.3 is proved.

3. Dynamical systems and the stop operator

The main part of this section is devoted to the projected dynamical system

$$\dot{w}(t) = Q_{\hat{w}(w)} (-F'(w(t))) , \quad w(0) = w_0 \in Z ,$$

(34)

where $F$ is a convex function as at the end of the previous section. The equilibrium point or stationary point of the projected dynamical system (34) is defined as the vector $w_\infty$ which fulfills the equation

$$\langle -F'(v), v - z \rangle \geq 0 \quad \forall z \in Z,$$

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The equivalence between the set $K_\infty$ of the equilibrium points of the projected dynamical system (34) and the set of set of global minima $K$ of the convex function $F$ is established in Lemma 2.3. This result permits the minimization of $F(w)$ with $w \in Z$ by solving the projected dynamical system (34) provided that a solution trajectory of the projected dynamical system (34) exists, is unique and converges asymptotically to the convex set $K_\infty$ of equilibrium points. But the projection transformation $Q_{\hat{w}(w)}$ introduces a discontinuity into the right-hand side of the differential equation (34) and the problem becomes difficult. Instead of trying to extend to our case the complicated methods which are described in [7], we transform the equation (34) in an operator-differential equation with a Lipschitz-continuous right-hand side. For this purpose we work with the space $AC(\mathbb{R};X)$ of absolutely continuous functions $u : \mathbb{R} \rightarrow X$, where $\mathbb{R}$ denotes the interval $[0,\infty]$. Keeping still fixed the convex closed set $Z \subset X$ from the previous section, we refer to [2,4] to recall that for every $u \in AC(\mathbb{R};X)$ and every initial condition $w_0 \in Z$ there exists a unique solution $w \in AC(\mathbb{R};X)$ to the variational inequality

$$\langle \ddot{u}(t) - \dot{w}(t), w(t) - z \rangle \geq 0 \quad \forall z \in Z , \quad (36)$$

where the dot denotes the derivatives with respect to $t$. The solution operator

$$S : Z \times AC(\mathbb{R};X) \rightarrow AC(\mathbb{R};X) : (w_0, u) \rightarrow w$$

(37)

is called the stop and its analytical properties have been systematically studied e.g. in Chapter 2 of [2]. We will need the following result which in principle goes back to [4].

Lemma 3.1 Let $u, w \in AC(\mathbb{R};X)$ be given, $w(t) \in Z$ for all $t \geq 0$, and let $S : Z \times AC(\mathbb{R};X) \rightarrow AC(\mathbb{R};X)$ be the stop. The following two conditions are equivalent.

(i) $\dot{w}(t) = Q_{\hat{w}(w)} (\ddot{u}(t))$ a.e.

(ii) $w = S[w(0), u]$.

Proof. We use the equivalence statement in Lemma 1.1. If (i) holds for some $t > 0$, then (17) reads

$$\langle \ddot{u}(t) - \dot{w}(t), w(t) - z \rangle \geq 0 \quad \forall z \in Z , \quad (38)$$

hence (ii) is fulfilled. Conversely, if (38) holds for some $t > 0$, then putting alternatively $z = w(t+h)$, $z = w(t-h)$ for $h \in [0,\delta]$, dividing (38) by $h$, and letting $h$ tend to $0^+$ we obtain

$$\langle \ddot{u}(t) - \dot{w}(t), \dot{w}(t) \rangle = 0 . \quad (39)$$

Moreover, for $y \in N(w(t))$ we have
and arguing similarly as above we obtain
\[ \langle y, \dot{w}(t) \rangle = 0, \]
(41)
hence \( \dot{w}(t) \in T(w(t)). \) From Lemma 2.1 and (38) - (41) we conclude that (i) holds.

Assume that (34) admits a solution \( w \in AC(\mathbb{R}_+; X) \) such that \( w(t) \in Z \) for all \( t \geq 0. \) For \( t \geq 0 \) we define an auxiliary function \( u \in AC(\mathbb{R}_+; X) \) by the formula
\[ u(t) = -\int_0^t F'(w(s)) ds. \]
(42)
From Lemma 3.1 it follows that \( w = S[w_0, u], \) and differentiating (42) we obtain the system
\[
\begin{align*}
(i) & \quad \dot{u}(t) + F'(w(t)) = 0, \\
(ii) & \quad w = S[w_0, u], \\
(iii) & \quad u(0) = 0.
\end{align*}
\]
(43)
By virtue of Lemma 3.1, problems (34) and (43) are equivalent. We will see that independently of the choice of \( Z, \) the stop \( S[w_0, \cdot] \) is for every \( T > 0 \) a Lipschitz continuous mapping from the restriction \( AC([0, T]; X) \) of \( AC(\mathbb{R}_+; X) \) onto the interval \([0, T]\) into the space \( C([0, T]; X) \) of continuous functions on \([0, T]. \) In order to simplify the presentation, we moreover assume that \( F' : X \to X \) is Lipschitz continuous. This enables us to treat the discontinuous problem (34) by the standard technique of continuous dynamical systems and construct the solution in a standard way by successive approximations.

**Proposition 3.2** Let \( Z \subset X \) be an arbitrary non-empty convex closed set and let \( F : U \to \mathbb{R} \) be a convex continuously differentiable function defined in a convex open set \( U \subset X \) such that \( Z \subset U. \) Assume furthermore that \( F' : X \to X \) is Lipschitz continuous. Then the system (43) admits a unique solution \( (u, w) \in AC(\mathbb{R}_+; X) \times AC(\mathbb{R}_+; X) \) such that \( \dot{u}/dt \) is continuous and \( \dot{w}/dt \in L_{loc}^\infty(\mathbb{R}_+; X). \)

**Proof.** We fix some final time \( T > 0, \) and for \( t \in [0, T] \) and \( n \in \mathbb{R} \) define recursively the sequences
\[ u^{(n)}(t) = o, \quad w^{(n)}(t) = S[w_0, u^{(n-1)}](t), \quad u^{(n)}(t) = \int_0^t F'(w^{(n-1)}(\tau)) d\tau. \]
(44)
From (36) it follows that
\[ \langle \dot{u}^{(n)}(t) - \dot{w}^{(n)}(t), -\dot{u}^{(n)}(t) + \dot{w}^{(n)}(t) \rangle \geq 0, \quad \forall n, m \in \mathbb{R} \cup \{0\}, \]
(45)
hence
\[ \frac{1}{2} \frac{d}{dt} \| w^{(n)}(t) - w^{(m)}(t) \|^2 \leq \| \dot{u}^{(n)}(t) - \dot{w}^{(m)}(t) \| \| w^{(n)}(t) - w^{(m)}(t) \| \quad \forall n, m \in \mathbb{R} \cup \{0\}, \]
or simply
\[ \frac{d}{dt} \| w^{(n)}(t) - w^{(m)}(t) \| \leq \| \dot{u}^{(n)}(t) - \dot{w}^{(m)}(t) \| \quad \forall n, m \in \mathbb{R} \cup \{0\}. \]
(46)
Integrating (46) yields
Let $L > 0$ be the Lipschitz constant of $F'$, that is,

$$
\|F'(x) - F'(y)\| \leq L \|x - y\| \quad \forall x, y \in U.
$$

By (44) and (47) we have for every $n \in \mathbb{N}$ and $t \in [0, T]$ that

$$
\left\| \dot{u}^{(n)}(t) - \dot{u}^{(n)}(t) \right\| \leq \frac{L}{n!} \int_0^t \left\| \dot{u}^{(n)}(\tau) - \dot{u}^{(n)}(\tau) \right\| d\tau.
$$

By induction over $n$ we obtain

$$
\left\| w^{(n)}(t) - w^{(n)}(t) \right\| \leq \frac{L}{n!} \int_0^t \left\| \dot{u}^{(n)}(\tau) - \dot{u}(\tau) \right\| d\tau \quad \forall t \in [0, T] \quad \forall n \in \mathbb{N},
$$

hence $w^{(n)}$ converge uniformly to $w$. Passing to the limit in (44) as $n \to \infty$ we check that $(u, w) \in AC([0, T]; X)$ is a solution to (43), hence in particular $d u/dt$ is continuous. From (39) it follows that $\|d w/dt\| \leq \|d u/dt\|$ a.e., hence $d w/dt \in L^\infty([0, T]; X)$. The uniqueness is easy: for two solutions $(u_i, w_i), i = 1, 2,$ we have

$$
\langle \dot{u}_1 - \dot{u}_2, w_1 - w_2 \rangle + \langle F'(w_1) - F'(w_2), w_1 - w_2 \rangle = 0 \quad \text{a.e.},
$$

where $\langle \dot{u}_1 - \dot{u}_2, w_1 - w_2 \rangle \geq \frac{1}{2} \|w_1 - w_2\|^2$ a.e., $\langle F'(w_1) - F'(w_2), w_1 - w_2 \rangle \geq 0$, hence $w_1 = w_2, u_1 = u_2$. We thus proved the existence of a unique solution to (43) on each interval $[0, T]$. The solution thus can be extended to $\mathbb{R}_+$ and the proof is complete.

The main result of this section reads as follows.

**Proposition 3.3** Assume that the set $K$ is non-empty, and let $u, w \in AC(\mathbb{R}_+, X)$ satisfy (43). Then there exists $w_\infty \in K$ such that

(i) $\lim_{t \to \infty} w(t) = w_\infty$.

(ii) There exists a nonincreasing function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|d w/dt\| \leq p(t)$ a.e. and $\lim_{t \to \infty} \sqrt{p(t)} = 0$.

(iii) $F(w(t)) \leq F(w(s))$ for all $t > s \geq 0$, and $\int_0^t \|w(t)\| dt = F(w_\infty) - F(w_\infty)$.

**Proof.** For $t > 0$ and $h > 0$ the solution satisfies the identity

$$
\dot{u}(t + h) - \dot{u}(t) + F'(w(t + h)) - F'(w(t)) = o.
$$

We take the scalar product of its left-hand side with $w(t + h) - w(t)$. From (36) it follows that
\[ \langle \dot{u}(t + h) - \dot{w}(t + h), w(t + h) - w(t) \rangle \geq 0 \quad \text{a.e.,} \]
\[ \langle \dot{u}(t) - \dot{w}(t), w(t) - w(t + h) \rangle \geq 0 \quad \text{a.e.,} \]

hence
\[ \langle \dot{u}(t + h) - \dot{u}(t), w(t + h) - w(t) \rangle \geq \langle \dot{w}(t + h) - \dot{w}(t), w(t + h) - w(t) \rangle . \]

The convexity of \( F \) implies that \( F' \) is monotone, that is,
\[ \langle F'(w(t + h)) - F'(w(t)), w(t + h) - w(t) \rangle \geq 0 . \]

The above considerations lead to the inequality
\[ \frac{d}{dt} \| w(t + h) - w(t) \| \leq 0 \quad \text{a.e.} \forall h > 0 , \tag{52} \]
hence
\[ \| w(t + h) - w(t) \| \leq \| w(s + h) - w(s) \| \quad \forall t > s \geq 0 \quad \forall h > 0 . \tag{53} \]

Letting \( h \) tend to \( 0^+ \) we obtain
\[ \| \dot{w}(t) \| \leq \| \dot{w}(s) \| \quad \text{for all Lebesgue points} \ t > s > 0 \quad \text{of} \ \dot{w} . \tag{54} \]

We now put \( p(t) = \sup \ \text{ess} \| d\omega(r)/dr \| ; \ r \in [t, \infty) \). Then \( p \) is nonincreasing, and the complement of the set \( A \) of all \( t > 0 \) which are Lebesgue points of \( d\omega/dt \) and continuity points of \( p \) is of measure zero. For all \( r, \ t \in A, \ r > t > 0 \), we have \( \| d\omega(r)/dr \| \leq \| d\omega(t)/dt \| \), hence \( p(t) \leq \| d\omega(t)/dt \| \). On the other hand, for \( t \in A \) and \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \| d\omega(t)/dt \| \leq p(s) \leq p(t) + \epsilon \) for \( s \in [t - \delta, t] \). Letting \( \epsilon \) tend to \( 0^+ \) we obtain \( \| d\omega(t)/dt \| = p(t) \) for all \( t \in A \). We further take the scalar product of (43) (i) with \( d\omega(t)/dt \). Then (39) yields
\[ \frac{d}{dt} \| w(t) \|^2 + \frac{d}{dt} F(w(t)) = 0 \quad \text{a.e.} \tag{55} \]
hence
\[ \int_0^t \| w(t) \|^2 \ dt + F(w(t^*)) - F(w_0) = 0 \quad \forall t^* > 0 , \tag{56} \]

which for \( t^* \to \infty \) yields
\[ \int_0^\infty p(t)^2 \ dt \leq F(w_0) - F_{\min} . \tag{57} \]

Assume that there exists \( c > 0 \) and a sequence \( 0 < t_1 < t_2 < \ldots \) such that \( \lim_{n \to \infty} t_n = \infty \) and \( p(t_n) \geq ct_n^{-1/2} \) for all \( n \in \mathbb{R} \). Then
\[ \int_0^\infty p(t)^2 \ dt \geq \sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} p(t)^2 \ dt \geq c^2 \sum_{n=1}^\infty (1 - \frac{t_n}{t_{n+1}}) . \]

Set \( a_n = 1 - t_n/t_{n+1} \) for \( n \in \mathbb{R} \). Then \( \log(t_{n+1}) = \log(t_n) - \log(1 - a_n) \), hence
\[ +\infty = \sum_{n=1}^\infty \log(1 - a_n) \leq \max_{n \in \mathbb{R}} \frac{\log(1 - a_n)}{-a_n} \sum_{n=1}^\infty a_n < +\infty \tag{58} \]

which is a contradiction, hence (ii) is verified. The next estimate consists in choosing an arbitrary \( \nu \in K \), and taking the scalar product of (43) (i) with \( w(t) - \nu \). Then
\[ \langle \dot{u}(t), w(t) - v \rangle + \langle F'(w(t)), w(t) - v \rangle = 0. \]  
(59)

We have \( \langle \dot{u}(t), w(t) - v \rangle \geq \langle w(t), w(t) - v \rangle \) by (36) and \( \langle F'(w(t)), w(t) - v \rangle \geq F(w(t)) - F(v) \) by (33), and from (59) it follows that

\[
\langle w(t), w(t) - v \rangle + F(w(t)) - F(v) \leq 0
\]

that is

\[
\frac{1}{2} \frac{d}{dt} \|w(t) - v\|^2 + F(w(t)) - F(v) \leq 0.
\]

(61)

We have in particular that

\[
\|w(t) - v\| \leq \|w(s) - v\| \quad \forall t > s \geq 0.
\]

(62)

We find a sequence \( t_n \to \infty \) and an element \( w_\infty \in Z \) such that

\[
w(t_n) \to 0, \quad w(t_n) \to w_\infty \quad \text{as} \quad n \to \infty.
\]

(63)

Inequality (60) for \( t = t_n \) and \( n \to \infty \) yields that \( F(w_\infty) \leq F(v) = F_{\min} \), hence \( w_\infty \in K \). This enables us to finish the proof. Choosing \( v = w_\infty \) in (62) we see that the whole trajectory of \( w \) converges to \( w_\infty \), hence (i) holds, while (iii) follows from (55) and (56) for \( t^* \to \infty \).

4. Conclusion

In control theory many identification problems with parameters which are subject to specific restrictions can be stated as convex programming problems. These problems can be solved on-line by the time-integration of a special time-invariant projected dynamical system with a discontinuous right-hand side. The paper develops an alternative formulation of this projected dynamical system based on a multidimensional stop operator. In contrast to the original formulation the new right-hand side is continuous and the problem is thus accessible to conventional analysis methods which easily give results on existence, uniqueness and stability properties of the corresponding solution trajectories. In future works the presented on-line identification method will be used as a part of an iterative compensation scheme for complex hysteretic nonlinearities.

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6. References